

A p -ADIC CONSTRUCTION OF ATR POINTS ON \mathbb{Q} -CURVES

XAVIER GUITART AND MARC MASDEU

ABSTRACT. In this note we consider certain elliptic curves defined over real quadratic fields isogenous to their Galois conjugate. We give a construction of algebraic points on these curves defined over almost totally real number fields. The main ingredient is the system of Heegner points arising from Shimura curve uniformizations. In addition, we provide an explicit p -adic analytic formula which allows for the effective, algorithmic calculation of such points.

1. INTRODUCTION

At the beginning of the 2000's Darmon introduced two constructions of local points on modular elliptic curves over number fields: the Stark–Heegner points [Dar01] and the ATR points [Dar04, Chapter 8]. Both types of points are expected to be algebraic and to behave in many aspects as the more classical Heegner points. Although the two constructions bear some formal resemblances, a crucial difference lies in the nature of the local field involved: the former is p -adic and the later is archimedean. In order to explain the importance of this distinction, let us briefly recall the constructions and some of the features that are currently known about them.

Let E be an elliptic curve defined over \mathbb{Q} of conductor N , and let K be a real quadratic field such that the sign of the functional equation of $L(E/K, s)$ is -1 . Let p be a prime that divides N exactly and that is inert in K . Under an additional Heegner-type hypothesis, Stark–Heegner points in $E(K_p)$ are constructed in [Dar01] by means of certain p -adic line integrals. They are conjecturally global and defined over narrow ring class fields of K . This was later generalized by Greenberg [Gre09] to the much broader setting in which E is defined over any totally real number field F , and K/F is any quadratic extension in which some prime divisor of the conductor of E is inert and $L(E/K, s)$ has sign -1 .

There is extensive numerical evidence in support of the rationality of such p -adic points (cf. [DG02], [DP06], [GM], [GM13]), but the actual proof in general seems to be still far out of reach. In spite of this, in some very special cases the Stark–Heegner points have been verified to be global. In these particular settings they coexist with Heegner points, and they can actually be seen to be related to them [BD09], [LV11]. The p -adic nature of the points seems to play a key role in these arguments, by means of the connection between the formal logarithm of the Stark–Heegner points and the special values of suitable p -adic L -functions (see also [BDP13], [DRa], [DRb], and [BDR]).

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The archimedean counterparts to these points, as introduced in [Dar04, Chapter 8], and later generalized by Gartner [Gär12], seem to be even more mysterious. The simplest and original setting is that of an elliptic curve E defined over a totally real number field F , and M/F a quadratic ATR extension (i.e., M is almost totally real, meaning that it has a single complex place). In this case the points are constructed by means of complex integrals and thus they lie in $E(\mathbb{C})$. They are also expected to be global, and there is some numerical evidence of it [DL03], [GM].

However, for the archimedean constructions it is not (the logarithm of) the points which is expected to be related to special values of complex L -functions, but their heights (very much in the spirit of Gross–Zagier formulas). It is this crucial difference with the p -adic case what seems to prevent any attempt of showing their rationality, even in the very particular instances in which they coexist with Heegner points. It could be arguably said that complex ATR points are much more difficult to handle than their p -adic counterparts. Thus, even in the simplest situations in which one wants to compare them with Heegner points in order to show their rationality, it is desirable to have p -adic constructions of such points at one's disposals.

In light of the above discussion, the goal of the present paper is to present a p -adic construction of algebraic points defined over ATR fields. To be more precise, we consider a real quadratic field F and a non-CM elliptic curve E/F that is F -isogenous to its Galois conjugate (this is sometimes referred to as a \mathbb{Q} -curve in the literature). Suppose that M/F is a quadratic ATR extension such that the sign of the functional equation of $L(E/M, s)$ is -1 . We describe a p -adic construction of algebraic points in $E(M)$, which are manufactured by means of suitable Heegner points in a certain Shimura curve parametrizing E .

The points that we construct are algebraic (for they essentially come from Heegner points in certain modular abelian varieties) and given in terms of p -adic line integrals. Observe that in this set up one can also consider p -adic Stark–Heegner points, e.g., the ones constructed by Greenberg [Gre09]. It would be very interesting to investigate the possible relationship between these two types of points.

The fact that our construction is given in terms of p -adic line integrals also has another consequence, which constitutes in fact one of the remarkable features of the construction: it gives rise to a completely explicit and efficient algorithm for computing the points.

Our construction is inspired by and builds on the work of Darmon–Rotger–Zhao [DRZ12]. In the next section we recall the points introduced in [DRZ12], and give an overview of the rest of the paper.

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2. BACKGROUND AND OUTLINE OF THE CONSTRUCTION

Our construction can be seen as a generalization of that of [DRZ12]. In order to put it in context, it is illustrative to examine first the case of elliptic curves over \mathbb{Q} . So let us (temporarily) denote by E an elliptic curve over \mathbb{Q} of conductor N . The Modularity Theorem [Wil95], [TW95], [BCDT01] provides a non-constant map

$$(2.1) \quad \pi_E: X_0(N) \longrightarrow E,$$

where $X_0(N)$ denotes the modular curve parametrizing cyclic isogenies $C \rightarrow C'$ of degree N . This moduli interpretation endows $X_0(N)$ with a canonical set of algebraic points known as CM or Heegner points which give rise, when projected under π_E , to a systematic construction of algebraic points on E .

To be more precise, suppose that K is a quadratic imaginary field and $\mathcal{O} \subset K$ is an order of discriminant coprime with N . In addition, suppose that K satisfies the *Heegner condition*:

(H). All the primes dividing N are split in K .

Under this assumption, there exist elliptic curves C and C' with complex multiplication by \mathcal{O} , together with a cyclic isogeny $C \rightarrow C'$ of degree N . The theory of complex multiplication implies that the point in $X_0(N)$ corresponding to $C \rightarrow C'$ is, in fact, algebraic and defined over the ring class field of \mathcal{O} .

Moreover, the corresponding Heegner point on E can be computed by means of the complex uniformization derived from (2.1) which, in view of the identifications $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$ and $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda_E$, is of the form

$$(2.2) \quad \pi_E: \Gamma_0(N) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathbb{C}/\Lambda_E.$$

The formula for computing the Heegner point corresponding to $C \rightarrow C'$ is then

$$(2.3) \quad \Phi_W \left(\int_{\tau}^{i\infty} 2\pi i f_E(z) dz \right),$$

where $f_E(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$ denotes the weight two newform for $\Gamma_0(N)$ whose L -function equals that of E , the map $\Phi_W: \mathbb{C}/\Lambda_E \rightarrow E(\mathbb{C})$ is the Weierstrass uniformization, and $\tau \in \mathcal{H} \cap K$ is such that $C \simeq \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $C' \simeq \mathbb{C}/\mathbb{Z} + N\tau\mathbb{Z}$.

This type of Heegner points are one of the main ingredients intervening in the proof of the Birch and Swinnerton-Dyer conjecture for curves over \mathbb{Q} of analytic rank ≤ 1 [GZ86],[Kol88]. Moreover, and perhaps more relevant to the purpose of the present note, the formula (2.3) is completely explicit and computable, as the Fourier coefficients a_n can be obtained by counting points on the several reductions of E mod p . In other words, (2.3) provides with an effective algorithm for computing points on E over abelian extensions of K , which turn out to be of infinite order whenever the analytic rank is 1. See, e.g., [Elk94] for a discussion of this method and examples of computations.

Suppose now that K does not satisfy the Heegner condition, and factor N as $N = N^+ N^-$, where N^+ contains the primes that split in K and $N^- > 1$ those that are inert. In this case there is a generalization of the above Heegner point construction, that works under the less restrictive *Heegner-Shimura condition*:

(H'). N^- is squarefree and the product of an even number of primes.

In this set up (2.1) is replaced by a uniformization of the form

$$(2.4) \quad \pi_E^{N^-}: X_0(N^+, N^-) \longrightarrow E,$$

where $X_0(N^+, N^-)$ is the Shimura curve of level N^+ associated to the indefinite quaternion algebra B/\mathbb{Q} of discriminant N^- . The moduli interpretation of $X_0(N^+, N^-)$, combined with the theory of complex multiplication, can also be used to construct Heegner points on E that are defined over ring class fields of orders $\mathcal{O} \subset K$.

There is also an analogue of formula (2.3), but in this case it seems to be much more difficult to compute in practice. In this case one needs to integrate modular forms associated to B and, since B is division, the Shimura curve $X_0(N^+, N^-)$ has no cusps. Therefore the corresponding modular forms do not admit Fourier expansions, which are the crucial tool that allow for the explicit calculation of (2.3). Elkies developed methods for performing such computations under some additional hypothesis [Elk98]. More recently Voight–Willis [VW14] using Taylor expansions and Nelson [Nel12] using the Shimizu lift have been able to compute some of these CM points.

In a different direction, there is an alternative method that allows for the numerical calculation of Heegner points associated to quaternion division algebras. The key idea is the use of the rigid analytic p -adic uniformization derived from (2.4), instead of the complex one. The Čerednik–Drinfel’d theorem provides a model for $X_0(N^+, N^-)$ as the quotient of the p -adic upper half plane \mathcal{H}_p by Γ , a certain subgroup in a definite quaternion algebra. Bertolini and Darmon [BD98], building on previous work of Gross [Gro87], give an explicit formula for the uniformization map

$$\Gamma \backslash \mathcal{H}_p \longrightarrow E(\mathbb{C}_p)$$

in terms of the so-called multiplicative p -adic line integrals of rigid analytic modular forms for Γ . Such integrals can be very efficiently computed, thanks to the methods of M. Greenberg [Gre06] (which adapt Pollack–Stevens’ overconvergent modular symbols technique [PS11]) and to the explicit algorithms provided by Franc–Masdeu [FM].

Let us not return to the setting that we consider in the present note. Namely, F is a real quadratic field and E/F is an elliptic curve without complex multiplication that is F -isogenous to its Galois conjugate. As a consequence of Serre’s modularity conjecture and results of Ribet E can be parametrized by a modular curve of the form $X_1(N)$, associated with the moduli problem of classifying elliptic curves C together with a point of order N in C . This property was exploited by Darmon–Rotger–Zhao in [DRZ12] in order to construct certain algebraic ATR points on E by means of Heegner points on $X_1(N)$. Let us briefly explain the structure of the construction.

Consider the uniformization mentioned above

$$(2.5) \quad \pi_E: X_1(N) \longrightarrow E,$$

where now N is a certain integer that is related to the conductor of E . Let us assume, for simplicity, that N is squarefree. We remark that π_E is defined over F . Let M/F be a quadratic extension that has one complex and two real places (this is what is known as an Almost Totally Real (ATR) extension, because it has exactly one complex place). There is a natural quadratic imaginary field K associated to M as follows: if $M = F(\sqrt{\alpha})$ for some $\alpha \in F$, then $K = \mathbb{Q}(\sqrt{Nm_{F/\mathbb{Q}}(\alpha)})$. Suppose that K satisfies the following Heegner-type condition, which might be called the Heegner–Darmon–Rotger–Zhao condition:

(DRZ). All the primes dividing N are split in K .

Under this assumption, the method presented in [DRZ12] uses Heegner points on $X_1(N)$ associated to orders in K to construct points in $E(M)$, which are shown to be of infinite order in situations of analytic rank one. One of the salient features

of this construction is that it is explicitly computable. In fact, there is a formula analogous to (2.3), giving the points as integrals of certain classical modular forms for $\Gamma_1(N)$.

In the first part of the paper, which consists of Sections 3 to 5, we extend the construction of [DRZ12] to the situation in which K satisfies the following, less restrictive, Heegner–Shimura-type condition:

(DRZ’). The number of prime divisors of N^- is even.

(As before we write $N = N^+N^-$, where N^+ contains the primes that split in K and N^- those that remain inert). As we will see, this condition is satisfied whenever $L(E/M, s)$ has sign -1 (see Proposition 3.4 below). In particular, it is satisfied when the analytic rank of E/M is 1.

The idea of our construction, inspired by the case of curves over \mathbb{Q} reviewed above, consists in replacing (2.5) by a uniformization of the form

$$(2.6) \quad \pi_E^{N^-} : X_1(N^+, N^-) \longrightarrow E,$$

where $X_1(N^+, N^-)$ is a suitable Shimura curve attached to an indefinite quaternion algebra B/\mathbb{Q} of discriminant N^- and level structure “of Γ_1 -type”. This main construction of ATR points in $E(M)$ is presented in §5, after developing some preliminary results. Namely, in §3 we briefly review \mathbb{Q} -curves and we prove some results in Galois theory that relate certain ring class fields of K with M , and in §4 we define the CM points on the Shimura curves that will play a role in our construction and determine their field of definition.

Just as in the classical case of curves over \mathbb{Q} , the CM points in $X_1(N^+, N^-)$, and hence the points that we construct in $E(M)$, are difficult to compute using the complex uniformization. Once again, the absence of cusps in $X_1(N^+, N^-)$ and thus the lack of Fourier coefficients makes it difficult to compute the integrals that appear in the explicit formula (cf. (5.4) below).

The second part of the article gives a p -adic version of the construction. As has been mentioned in the introduction, this might be useful in order to relate it to p -adic Stark–Heegner points. Another advantage of this p -adic construction is that it is explicitly computable. Concretely, in §6 we exploit the p -adic uniformization of $X_1(N^+, N^-)$ given by the Čerednik–Drinfel’d theorem and the explicit uniformization of Bertolini–Darmon in terms of multiplicative p -adic integrals. Combining this with a slight generalization of the algorithms of Franc–Masdeu [FM], our construction provides an efficient algorithm for computing algebraic ATR points in \mathbb{Q} -curves. We conclude with an explicit example of such computation in §7.

3. \mathbb{Q} -CURVES AND ATR EXTENSIONS

In this section we recall some basic facts on \mathbb{Q} -curves and their relation with classical modular forms for $\Gamma_1(N)$. We also give some preliminary results on certain Galois extensions associated to ATR fields that will be needed in the subsequent sections, as they will be related to the field of definition of the Heegner points under consideration.

3.1. \mathbb{Q} -curves and modular forms. Let F be a real quadratic field and let E/F be an elliptic curve that is F -isogenous to its Galois conjugate. Such a curve is sometimes referred in the literature as a *\mathbb{Q} -curve completely defined over F* . Let $\mathfrak{N}_E \subseteq F$

denote the conductor of E , which for simplicity we assume to be squarefree and relatively prime to $\text{disc}(F/\mathbb{Q})$.

Under these hypotheses, \mathfrak{N}_E is generated by a rational integer (cf. the discussion on [GJG10]), say

$$\mathfrak{N}_E = N_0 \mathcal{O}_F \text{ for some } N_0 \in \mathbb{Z}_{\geq 1}.$$

As a consequence of Serre's modularity conjecture and results of Ribet [Rib92], there exists a classical elliptic modular form $f = f_E \in S_2(N, \psi)$ whose field of Fourier coefficients is a quadratic number field K_f and such that

$$L(E/F, s) = \prod_{\sigma: K_f \hookrightarrow \mathbb{C}} L(f^\sigma, s),$$

for some integer N and some quadratic character ψ of conductor N_ψ dividing N . For simplicity, we assume that N is odd and squarefree.

In order to rule out some trivial cases we will assume also that E is not the base change of a curve over \mathbb{Q} . Slightly more generally, since the arithmetic problems that we are interested in are in fact invariant under isogeny, we can assume that E is not F -isogenous to a curve over \mathbb{Q} . Then the restriction of scalars $\text{Res}_{F/\mathbb{Q}} E$ is simple over \mathbb{Q} and isogenous to A_f , the modular abelian variety attached to f by the Eichler–Shimura construction. Moreover, $\mathbb{Q} \otimes \text{End}_{\mathbb{Q}}(A_f)$ is isomorphic to K_f and, by results of Carayol [Car89], the conductor of A_f is equal to N^2 .

The most interesting situation for the arithmetic applications of the present note is when K_f is a quadratic imaginary field, which we assume from now on. This implies that ψ is not trivial, and that it is the character corresponding by class field theory to the extension F/\mathbb{Q} . In particular, by the conductor-discriminant formula we have that $N_\psi = \text{disc}(F/\mathbb{Q})$. Combining the formula for the conductor of the restriction of scalars [Mil72], with the fact that A_f has conductor N^2 we obtain that

$$N = N_0 N_\psi.$$

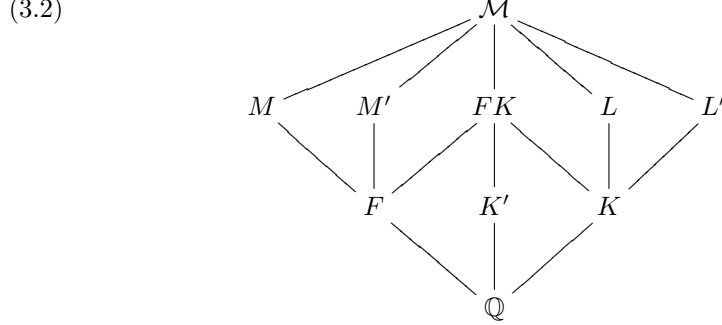
3.2. ATR extensions. Let M/F be a quadratic almost totally real (ATR) extension of discriminant prime to \mathfrak{N}_E and such that the L -function $L(E/M, s)$ has sign -1 . This condition is equivalent (see, e.g., the discussion of [Dar04, §3.6]) to the set

$$(3.1) \quad \{\mathfrak{p} \mid \mathfrak{N}_E : \mathfrak{p} \text{ is inert in } M\}$$

having even cardinality.

We have that $M = F(\sqrt{\alpha})$ for some $\alpha \in F$. We set $M' = F(\sqrt{\alpha'})$, where α' stands for the Galois conjugate of α . Then $\mathcal{M} = MM'$ is the Galois closure of M and its Galois group $\text{Gal}(\mathcal{M}/\mathbb{Q})$ is isomorphic to $D_{2,4}$, the dihedral group of 8

elements. The diagram of subfields of \mathcal{M} is of the form



where $K = \mathbb{Q}(\sqrt{\alpha\alpha'})$. Observe that K is a quadratic imaginary field, for M is ATR and necessarily $\alpha\alpha' = \text{Nm}_{F/\mathbb{Q}}(\alpha) < 0$. From now on, we will assume that the discriminant of K is relatively prime to N .

We will see that all the primes dividing N_ψ are split in K (see Lemma 3.2 below). We consider a decomposition of N of the form $N = N^+ N^-$, where

- $N^+ = N_\psi N_0^+$, and N_0^+ is the product of primes $\ell \mid N_0$ such that ℓ is split in K , and
- N^- is the product of $\ell \mid N_0$ such that ℓ is inert in K .

As we already mentioned in the Introduction, one of the central ideas of [DRZ12] is that Heegner points on A_f can be used to manufacture points on $E(M)$. Indeed, an explicit such construction is provided in [DRZ12, §4], under the assumption that $\mathfrak{N}_E = (1)$. Such construction, in fact, is easily seen to be valid under the following slightly more general Heegner-type condition:

(DRZ). $N^- = 1$ (i.e., all the primes dividing N are split in K).

Let us briefly review the structure of the construction in this case (we refer to [DRZ12] for the details). Let us (temporarily) denote by $\Gamma_0(N)$ the subgroup of $\text{SL}_2(\mathbb{Z})$ of upper triangular matrices modulo N , and by $\Gamma_\psi(N)$ the congruence subgroup

$$\Gamma_\psi(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : N \mid c, \psi(a) = 1 \right\} \subset \Gamma_1(N).$$

Let $X_0(N)$ (resp. $X_\psi(N)$) denote the modular curve associated to $\Gamma_0(N)$ (resp. to $\Gamma_\psi(N)$), and let $J_0(N)$ (resp. $J_\psi(N)$) denote its Jacobian. The variety A_f/\mathbb{Q} turns out to be a quotient of $J_\psi(N)/\mathbb{Q}$. Since A_f is isogenous over F to E^2 , it follows that E admits a morphism (defined over F) from $J_\psi(N)$. Therefore we obtain a uniformization

$$(3.3) \quad J_\psi(N) \longrightarrow E$$

which is defined over F .

On the other hand, the inclusion $\Gamma_\psi(N) \subset \Gamma_0(N)$ induces a degree 2 map $X_\psi(N) \rightarrow X_0(N)$, and the Heegner points in $X_\psi(N)$ are the preimages of the Heegner points in $X_0(N)$. Denote by $M_0(N) \subset \text{M}_2(\mathbb{Z})$ the set of matrices which are upper triangular modulo N . An embedding $\varphi: K \hookrightarrow M_2(\mathbb{Q})$ is said to be of conductor c and level N if $\varphi^{-1}(M_0(N))$ is equal to \mathcal{O}_c , the order of conductor c . The Heegner points in $X_0(N)$ associated to \mathcal{O}_c are in one to one correspondence with the optimal embeddings of level N and conductor c . They are defined over

its ring class field H_c , so that their preimages in $X_\psi(N)$ are defined over a certain quadratic extension L_c of H_c . This gives rise to Heegner points in $J_\psi(N)$ defined over L_c .

One of the results proved in [DRZ12] is that, for suitable choices of c , L_c contains L . Taking the trace from L_c down to L one obtains a point in $J_\psi(N)(L)$. Summing it with its conjugate by an appropriate element in $\text{Gal}(\mathcal{M}/\mathbb{Q})$ produces a point on $J_\psi(N)(M)$. Finally, projecting to E via (3.3) yields the point on $E(M)$.

The reason why the construction outlined above only works under the hypothesis that $N^- = 1$ is that, otherwise, there do not exist optimal embeddings $\varphi: K \hookrightarrow M_2(\mathbb{Q})$ of conductor c and level N . That is to say, there are no Heegner points in $X_0(N)$ defined over ring class fields of K .

The main goal of the present article is to provide a construction of Heegner points on $E(M)$ in the case $N^- > 1$. For that purpose, and similarly to the classical case of Heegner points on curves over \mathbb{Q} , we need to consider Heegner points coming from Shimura curves attached to division quaternion algebras. In the next section we introduce the Shimura curves that will play the role of $X_\psi(N)$ in our construction, and we discuss Heegner points on them.

Before that, we state some Galois properties of the fields in Diagram (3.2) and about certain number fields L_c , attached to orders in K of conductor c that will be the fields of definition of Heegner points. We also introduce some more notation that will be in force for the rest of the article.

3.3. Galois properties and the number of primes dividing N^- . In this subsection we study those properties of the field diagram (3.2) that are needed later. Let

$$\chi_M, \chi'_M: G_F \longrightarrow \{\pm 1\}$$

denote the quadratic characters of $G_F = \text{Gal}(\overline{\mathbb{Q}}/F)$ cutting out the extensions M and M' , respectively. Observe that we can, and often do, view them as characters on the ideles \mathbb{A}_F^\times . Similarly we define the characters

$$\chi_L, \chi'_L: G_K \longrightarrow \{\pm 1\},$$

and view them as characters of \mathbb{A}_K^\times . We also denote by ε_F and ε_K the quadratic characters on $\mathbb{A}_{\mathbb{Q}}^\times$ corresponding to F and K , and by

$$\text{Nm}_{\mathbb{Q}}^F: \mathbb{A}_F^\times \longrightarrow \mathbb{Q}^\times, \quad \text{Nm}_{\mathbb{Q}}^K: \mathbb{A}_K^\times \longrightarrow \mathbb{Q}^\times$$

the norms on the ideles. Observe that, as remarked above, F is the field cut out by ψ . This means that, in fact, $\varepsilon_F = \psi$.

We will make use of the following properties of Diagram (3.2), which are given in Proposition 3.2 of [DRZ12].

- Lemma 3.1.** (1) $\chi_M \cdot \chi'_M = \varepsilon_K \circ \text{Nm}_{\mathbb{Q}}^F$ and $\chi_L \cdot \chi'_L = \varepsilon_F \circ \text{Nm}_{\mathbb{Q}}^K$.
(2) The central character of χ_M and χ'_M is ε_K , and the central character of χ_L and χ'_L is ε_F .
(3) $\text{Ind}_F^{\mathbb{Q}} \chi_M = \text{Ind}_K^{\mathbb{Q}} \chi_L$.

Let $\mathfrak{d}_{L/K}$ denote the discriminant of the extension L/K , which by the conductor-discriminant formula is the conductor of χ_L .

Lemma 3.2. *There exists a canonical ideal $\mathfrak{N}_\psi \subset \mathcal{O}_K$ of norm N_ψ . In particular, all primes dividing N_ψ are split in K .*

Proof. From the equality $\text{Ind}_F^{\mathbb{Q}} \chi_M = \text{Ind}_K^{\mathbb{Q}} \chi_L$, using the formula for the conductor of induced representations and the conductor-discriminant formula, we obtain

$$N_\psi \cdot \text{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{M/F}) = \text{disc}(K) \cdot \text{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K}).$$

This easily implies the existence of an ideal of norm N_ψ . The fact that it is canonical follows from our assumption that N (and hence N_ψ) is squarefree. Finally, all the primes dividing N_ψ are split in K by our running assumption that the discriminant of K is relatively prime to N . \square

Proposition 3.3. *The discriminant of L/K factorizes as $\mathfrak{d}_{L/K} = c\mathfrak{N}_\psi$, where c belongs to \mathbb{Z} and is relatively coprime to \mathfrak{N}_ψ .*

Proof. We consider primes p dividing $\text{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K})$.

Suppose that $p \mid N_\psi$. Then, in view of the previous lemma, p splits in K , say as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Then $\mathcal{O}_{K,\mathfrak{p}} \simeq \mathbb{Z}_p$, and by part (2) of Lemma 3.1 the composition

$$\mathbb{Z}_p^\times \longrightarrow \mathcal{O}_{K,\mathfrak{p}}^\times \times \mathcal{O}_{K,\bar{\mathfrak{p}}}^\times \xrightarrow{\chi_{L,\mathfrak{p}} \cdot \chi_{L,\bar{\mathfrak{p}}}} \{\pm 1\}$$

is equal to ψ_p (the local component of ψ at p), which is non trivial because $p \mid N_\psi$. But since $\chi_{L,\mathfrak{p}}, \chi_{L,\bar{\mathfrak{p}}}$ are quadratic characters, then necessarily exactly one them is trivial, say $\chi_{L,\bar{\mathfrak{p}}} = 1$ and $\chi_{L,\mathfrak{p}} \neq 1$. Then \mathfrak{p} divides exactly the conductor of χ_L (which is equal to $\mathfrak{d}_{L/K}$), and $\bar{\mathfrak{p}}$ does not divide it.

Now suppose that $p \nmid N_\psi$. That is to say, ψ_p is trivial. Let $\mathfrak{p} \mid p$ be a prime in K such that \mathfrak{p}^e divides exactly the conductor of χ_L . Then p necessarily splits in K , because otherwise the composition

$$\mathbb{Z}_p^\times \longrightarrow \mathcal{O}_{K,\mathfrak{p}}^\times \xrightarrow{\chi_{L,\mathfrak{p}}} \{\pm 1\}$$

would equal ψ_p , which is trivial. But this would contradict the fact that $\chi_{L,\mathfrak{p}}$ is non trivial (and restricted to \mathbb{Z}_p^\times would also be non trivial). Therefore, it must be $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ and $\chi_{L,\bar{\mathfrak{p}}} \simeq \chi_{L,\mathfrak{p}}^{-1}$, so that $\bar{\mathfrak{p}}^e$ also divides exactly the conductor of χ_L . \square

The Heegner points that we will use in our construction arise from a Shimura curve associated to an indefinite algebra of discriminant N^- . Therefore, the following result is key to our purposes.

Proposition 3.4. *The number of primes dividing N^- is even.*

Proof. Recall that $\mathfrak{N}_E = N_0\mathcal{O}_F$ and that the set

$$(3.4) \quad \{\mathfrak{p} \mid \mathfrak{N}_E : \mathfrak{p} \text{ is inert in } M\}$$

has even cardinality thanks to our running assumption that $L(E/M, s)$ has sign -1 . Every prime in the set (3.4) is above a prime $p \mid N_0$. Thus, in order to prove the proposition it is enough to prove the following claims.

Claim 1. Every prime $p \mid N_0^+$ gives rise to either zero or two primes in the set (3.4).

Claim 2. Every prime $p \mid N_0^- = N^-$ gives rise to exactly one prime in the set (3.4).

Proof of Claim 1. Let p be a prime dividing N_0^+ . Namely, p is a prime divisor of N_0 that splits in K . Observe that p can not ramify in F because $(N_\psi, N_0) = 1$. If p splits in F , say $p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}'$, by part (1) of Lemma 3.1 we see that

$$\chi_M(\mathfrak{p}) \cdot \chi_M(\mathfrak{p}') = \chi_M(\mathfrak{p}) \cdot \chi'_M(\mathfrak{p}) = \varepsilon_K(\mathrm{Nm}_{\mathbb{Q}}^F(\mathfrak{p})) = \varepsilon_K(p) = 1,$$

so that either both \mathfrak{p} and \mathfrak{p}' are inert in M , or both are split. In other words, either \mathfrak{p} and \mathfrak{p}' belong to (3.4), or none of them does.

If p remains inert in F , by part (2) of Lemma 3.1 we have that

$$\chi_M(p\mathcal{O}_F) = \varepsilon_K(p) = 1,$$

which means that $p\mathcal{O}_F$ is split in M , so that it does not belong to (3.4).

Proof of Claim 2. Let p be a prime dividing $N_0^- = N^-$. Again there are two possibilities.

- (1) If p is split in F , say $p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}'$, then by part (1) of Lemma 3.1 we have that

$$\chi_M(\mathfrak{p})\chi_M(\mathfrak{p}') = \chi_M(\mathfrak{p})\chi'_M(\mathfrak{p}) = \varepsilon_K(\mathrm{Nm}_{\mathbb{Q}}^F(\mathfrak{p})) = \varepsilon_K(p) = -1,$$

so exactly one of the primes above p is inert in M and therefore belongs to (3.4).

- (2) If p is inert in F , then by part (2) of Lemma 3.1 we see that

$$\chi_M(p\mathcal{O}_F) = \varepsilon_K(p) = -1,$$

and so $p\mathcal{O}_F$ is inert in M .

□

3.4. The field L_c . The aim of this subsection is to define a certain extension L_c of L , associated to ψ and to the order of conductor c in K . It will turn out to be the field of definition of the Heegner points that we will consider in Section 4.

Recall that from Proposition 3.3 the discriminant of L/K factorizes as

$$\mathfrak{d}_{L/K} = c\mathfrak{N}_\psi,$$

where c is a rational integer with $(c, N) = 1$ and \mathfrak{N}_ψ is an ideal in K of norm N_ψ . Let $N^+ = N_\psi N_0^+$ and let \mathfrak{N}^+ be an ideal of K of norm N^+ , such that $\mathfrak{N}_\psi \mid \mathfrak{N}^+$. We remark that such ideal is not canonical, but we fix a choice of \mathfrak{N}^+ throughout the paper, and we denote by $\bar{\mathfrak{N}}^+$ its complex conjugate.

Let H_c/K be the ring class field of K of conductor c . Denote by \mathbb{A}_K the adeles of K , and by $\hat{\mathcal{O}}_K = \prod_{\mathfrak{p}} \mathcal{O}_{K, \mathfrak{p}} \subset \mathbb{A}_{K, \text{fin}}$. The reciprocity map of class field theory provides an identification $\mathrm{Gal}(H_c/K) \simeq \mathbb{A}_K^\times / (K^\times U_c)$, where

$$U_c = \hat{\mathbb{Z}}^\times (1 + c\hat{\mathcal{O}}_K) \mathbb{C}^\times \subset \mathbb{A}_K^\times.$$

Following [DRZ12, §4.1] we define

$$U_c^0 = \{\alpha \in U_c : (\alpha)_{\mathfrak{N}^+} \in \ker(\psi) \subset (\mathbb{Z}/N^+\mathbb{Z})^\times\},$$

$$\bar{U}_c^0 = \{\alpha \in U_c : (\alpha)_{\bar{\mathfrak{N}}^+} \in \ker(\psi) \subset (\mathbb{Z}/N^+\mathbb{Z})^\times\}.$$

Here we are using the fact that \mathfrak{N}^+ has norm N^+ , so that we have isomorphisms

$$\mathcal{O}_{\mathfrak{N}^+}^\times / (1 + \mathfrak{N}^+ \mathcal{O}_{\mathfrak{N}^+}) \simeq (\mathbb{Z}/N^+\mathbb{Z})^\times,$$

$$\mathcal{O}_{\bar{\mathfrak{N}}^+}^\times / (1 + \bar{\mathfrak{N}}^+ \mathcal{O}_{\bar{\mathfrak{N}}^+}) \simeq (\mathbb{Z}/N^+\mathbb{Z})^\times.$$

Let L_c and L'_c be the fields corresponding by class field theory to U_c^0 and \bar{U}_c^0 respectively. That is to say

$$(3.5) \quad \text{Gal}(L_c/K) \simeq \mathbb{A}_K^\times / (K^\times U_c^0), \quad \text{Gal}(L'_c/K) \simeq \mathbb{A}_K^\times / (K^\times \bar{U}_c^0).$$

Both L_c and L'_c are quadratic extensions of H_c , and we denote by \tilde{H}_c the biquadratic extension of K given by $\tilde{H}_c = L_c L'_c$.

Lemma 3.5. *If c is the one given by Lemma 3.3, then L is contained in L_c . Therefore \mathcal{M} is contained in \tilde{H}_c .*

Proof. By class field theory it is enough to show that U_c^0 is contained in $\ker \chi_L$. Recall that the conductor of χ_L is equal to $\mathfrak{d}_{L/K}$ and hence equal to $c\mathfrak{N}_\psi$, with $\mathfrak{N}_\psi \mid \mathfrak{N}^+$. This means that χ_L factors through a character

$$(3.6) \quad \chi_L: \mathcal{O}_{K,c\mathfrak{N}^+}^\times / (1 + c\mathfrak{N}^+ \mathcal{O}_{K,c\mathfrak{N}^+}) \longrightarrow \{\pm 1\}.$$

Let (α) be a finite idele of K that belongs to U_c^0 . We aim to see that $\chi_L(\alpha) = 1$. Since α belongs to U_c , we can write it as $\alpha = a(1 + cx)$ for some $a \in \hat{\mathbb{Z}}$ and some $x \in \hat{\mathcal{O}}_K$. Locally, we can express this as

$$\alpha = a(1 + cx) = a \prod_{\mathfrak{p} \nmid c\mathfrak{N}^+} x_{\mathfrak{p}} \prod_{\mathfrak{p} \mid c} (1 + \mathfrak{p}^{v_{\mathfrak{p}}(c)} x_{\mathfrak{p}}) \prod_{\mathfrak{p} \mid \mathfrak{N}^+} x_{\mathfrak{p}}.$$

By (3.6) we see that

$$\chi_L \left(\prod_{\mathfrak{p} \nmid c\mathfrak{N}^+} x_{\mathfrak{p}} \prod_{\mathfrak{p} \mid c} (1 + \mathfrak{p}^{v_{\mathfrak{p}}(c)} x_{\mathfrak{p}}) \right) = 1.$$

Therefore, we see that

$$\chi_L(\alpha) = \chi_L \left(a \prod_{\mathfrak{p} \mid \mathfrak{N}^+} x_{\mathfrak{p}} \right) = \chi_L \left(\prod_{\mathfrak{p} \mid \mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}} \right).$$

Since \mathfrak{N}^+ has norm N^+ , which is squarefree, the idele $\prod_{\mathfrak{p} \mid \mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}}$ can be viewed as an element in $\mathbb{A}_{\mathbb{Q}}^\times$. Since $\chi_{L|\mathbb{A}_{\mathbb{Q}}^\times} = \psi$, we have that

$$\chi_L(\alpha) = \psi \left(\prod_{\mathfrak{p} \mid \mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}} \right) = 1,$$

where the last equality follows from the definition of U_c^0 . \square

4. CM POINTS ON SHIMURA CURVES WITH QUADRATIC CHARACTER

In this section we recall some basic facts and well-known properties of Shimura curves. We also introduce the CM points that will play a key role in our construction of points in $E(M)$ later in Section 5, and we use Shimura's reciprocity law to deduce their field of definition.

Let \mathcal{B}/\mathbb{Q} be the quaternion algebra of discriminant N^- . Thanks to Proposition 3.4 we see that \mathcal{B} is indefinite so we can, and do, fix an isomorphism

$$\iota_\infty: \mathcal{B} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R}).$$

Choose $\mathcal{R}_0 = \mathcal{R}_0(N^+, N^-)$ an Eichler order of level N^+ in \mathcal{B} together with, for every prime $\ell \mid N^+$, an isomorphism

$$\iota_\ell: \mathcal{B} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\simeq} M_2(\mathbb{Q}_\ell)$$

such that

$$\iota_\ell(\mathcal{R}_0) \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_\ell) : c \in \ell\mathbb{Z}_\ell \right\}.$$

In this way we also obtain an isomorphism

$$\iota_{N^+}: \mathcal{R}_0 \otimes \mathbb{Z}_{N^+} \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_{N^+}) : c \in N^+\mathbb{Z}_{N^+} \right\},$$

where $\mathbb{Z}_{N^+} = \prod_{p \mid N^+} \mathbb{Z}_p$. Let $\eta: \mathcal{R}_0 \rightarrow \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+}$ be the map that sends γ to the upper left entry of $\iota_{N^+}(\gamma)$ taken modulo N^+ . The character ψ can be regarded in a natural way as a character $\psi: \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+} \rightarrow \{\pm 1\}$. Let $\mathcal{U}_0 = \mathcal{R}_0^\times$ be the group of units in \mathcal{R}_0 , and define

$$(4.1) \quad \mathcal{U}_\psi = \{\gamma \in \mathcal{U}_0 : \psi \circ \eta(\gamma) = 1\}.$$

Let also Γ_0 (resp. Γ_ψ) denote the subgroup of norm 1 elements in \mathcal{U}_0 (resp. \mathcal{U}_ψ).

4.1. Shimura curves. Let $X_0 = X_0(N^+, N^-)$ be the Shimura curve associated to Γ_0 . Similarly, let $X_\psi = X_\psi(N^+, N^-)$ be the Shimura curve associated to Γ_ψ . See [BC91, Chapitre III] for the precise definitions. They are curves over \mathbb{Q} , whose complex points can be described as

$$(4.2) \quad X_0(\mathbb{C}) \simeq \Gamma_0 \backslash \mathcal{H}, \quad X_\psi(\mathbb{C}) \simeq \Gamma_\psi \backslash \mathcal{H},$$

where \mathcal{H} denotes the complex upper half plane, and Γ_0 and Γ_ψ act on \mathcal{H} via ι_∞ . The inclusion $\Gamma_\psi \subset \Gamma_0$ induces a degree 2 homomorphism defined over \mathbb{Q}

$$\pi_\psi: X_\psi \longrightarrow X_0.$$

4.2. CM points. Let c be an integer relatively prime to N and to the discriminant of K , and let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ be the order of conductor c in K . An algebra embedding $\varphi: \mathcal{O}_c \hookrightarrow \mathcal{R}_0$ is said to be an *optimal embedding of conductor c* if $\varphi(K) \cap \mathcal{R}_0 = \varphi(\mathcal{O}_c)$. Recall also the ideal $\mathfrak{N}^+ \subset K$ of norm N^+ that we fixed in §3.4, and that we denote by $\bar{\mathfrak{N}}^+$ its complex conjugate.

Definition 4.1. We say that an optimal embedding $\varphi: \mathcal{O}_c \hookrightarrow \mathcal{R}_0$ is normalized with respect to \mathfrak{N}^+ if it satisfies that

- (1) $\iota_\infty(\varphi(a)) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = a \begin{pmatrix} \tau \\ 1 \end{pmatrix}$ for all $a \in \mathcal{O}_c$ and all $\tau \in \mathbb{C}$ (here we view $K \subset \mathbb{C}$);
- and
- (2) $\ker(\eta \circ \varphi) = \mathfrak{N}^+.$

We denote by $\mathcal{E}(c, \mathcal{R}_0)$ the set of normalized embeddings with respect to \mathfrak{N}^+ .

The groups Γ_0 and Γ_ψ act on $\mathcal{E}(c, \mathcal{R}_0)$ by conjugation, and we denote by $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ and $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$ the corresponding (finite) sets of conjugacy classes. Each $\varphi \in \mathcal{E}(c, \mathcal{R}_0)$ has a unique fixed point τ_φ in \mathcal{H} . The image of τ_φ in $\Gamma_0 \backslash \mathcal{H} \simeq X_0(\mathbb{C})$ (resp. in $\Gamma_\psi \backslash \mathcal{H} \simeq X_\psi(\mathbb{C})$) only depends on the class of φ in $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ (resp. $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$). We will denote the point defined by τ_φ in the Shimura curve by $[\tau_\varphi]$. The points obtained in this way are the so-called *CM points* or *Heegner points*.

We denote by $\text{CM}_0(c)$ the set of *CM points of conductor c* corresponding to optimal embeddings normalized with respect to \mathfrak{N}^+ . That is to say

$$\text{CM}_0(c) = \{[\tau_\varphi] \in X_0(\mathbb{C}) : \varphi \in \mathcal{E}(c, \mathcal{R}_0)/\Gamma_0\}.$$

Similarly, we denote by $\text{CM}_\psi(c)$ their preimage under π_ψ , which can be described as

$$\text{CM}_\psi(c) = \{[\tau_\varphi] \in X_\psi(\mathbb{C}) : \varphi \in \mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi\}.$$

From now on we identify $\text{CM}_0(c)$ with $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ and $\text{CM}_\psi(c)$ with $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$. Every element in $\text{CM}_0(c)$ has two preimages in $\text{CM}_\psi(c)$, which are interchanged by the action of any element $W_\psi \in \Gamma_0 \setminus \Gamma_\psi$.

There is an action \star of \hat{K}^\times on $\mathcal{E}(c, \mathcal{R}_0)$, given as follows. For any $x = (x_{\mathfrak{p}})_{\mathfrak{p}} \in \hat{K}^\times$ and $\varphi \in \mathcal{E}(c, \mathcal{R}_0)$, the fractional ideal $\hat{\varphi}(x)\hat{\mathcal{R}}_0 \cap \mathcal{B}$ is principal, say generated by $\gamma_x \in \mathcal{B}^\times$. Let $a_x = \hat{\varphi}(x_{\mathfrak{N}^+})^{-1}\gamma_x$. Observe that $a_{x, \mathfrak{p}} \in \mathcal{R}_0^\times$ for every $\mathfrak{p} \mid \mathfrak{N}^+$, and therefore it makes sense to consider $\psi \circ \eta(a_x)$. Modifying each $\gamma_{x, \mathfrak{p}}$ by a unit if necessary, we can assume that γ_x is chosen in such a way that $\psi \circ \eta(a_x) = 1$. That is to say, $\hat{\varphi}(x_{\mathfrak{N}^+})^{-1}\gamma_x$ lies in the kernel of $\psi \circ \eta$. Then $x \star \varphi$ is defined as $x \star \varphi := \gamma_x^{-1}\varphi\gamma_x$.

By results of Shimura CM points are defined over K^{ab} , the maximal abelian extension of K . The Galois action on them is given in terms of the reciprocity map of class field theory

$$\text{rec}: \hat{K}^\times/K^\times \longrightarrow \text{Gal}(K^{ab}/K)$$

by means of *Shimura's reciprocity law*:

$$(4.3) \quad \text{rec}(x)^{-1}([\tau_\varphi]) = [\tau_{x \star \varphi}].$$

Here the action in the left is the usual Galois action on the $\overline{\mathbb{Q}}$ -points of a variety defined over \mathbb{Q} . One of its well known consequences is that $\text{CM}_0(c) \subset X_0(H_c)$, i.e. CM points of conductor c on X_0 are defined over the ring class field of conductor c . One can also derive from it the field of definition of $\text{CM}_\psi(c)$, which is precisely the field L_c defined in §3.4.

Proposition 4.2. $\text{CM}_\psi(c) \subset X_\psi(L_c)$.

Proof. It follows directly from (4.3) and the fact that U_c^0 acts trivially on $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$. \square

5. ATR POINTS ON \mathbb{Q} -CURVES

In this section we introduce the main construction of this note, namely an ATR point in E manufactured by means of CM points on X_ψ . To this end, let us briefly recall the setting of Section 3 and some of the results encountered so far. The curve E/F is a \mathbb{Q} -curve over the real quadratic field F . The field M a quadratic ATR extension of F such that $L(E/M, s)$ has sign -1 . This gives rise to a quadratic imaginary extension K , sitting in the field diagram (3.2). By the modularity theorem there exists a classical newform $f = f_E \in S_2(N, \psi)$ characterized by the equality of L -functions

$$L(E/F, s) = \prod_{\sigma: K_f \hookrightarrow \mathbb{C}} L(f, s).$$

The level N factorizes as $N = N^+ N^-$, where N^- is the product of primes dividing N which are inert in K (there are an even number of them). By Lemma 3.3 the discriminant of L/K factorizes as $c\mathfrak{N}_\psi$, with $c \in \mathbb{Z}$ and $\mathfrak{N}_\psi \subset K$ an integral ideal of norm N_ψ . Recall also that we fixed an ideal \mathfrak{N}^+ of norm N^+ with $\mathfrak{N}_\psi \mid \mathfrak{N}^+$. Recall also $\text{CM}_\psi(c)$, the set of Heegner points of conductor c (and normalized with respect to \mathfrak{N}^+), which lie in $X_\psi(L_c)$.

5.1. Construction of the ATR point. Next, we describe how to attach to each degree 0 divisor $D \in \text{Div}^0 \text{CM}_\psi(c)$ a point $P_D \in E(M)$. Let $S_2(\Gamma_\psi) = S_2(\Gamma_\psi(N^+, N^-))$ denote the space of weight two newforms with respect to Γ_ψ . Thanks to the Jacquet–Langlands correspondence there exists a newform $g \in S_2(\Gamma_\psi)$ such that $L(g, s) = L(f, s)$. In other words, g has the same system of eigenvalues by the Hecke operators as f . In addition, if we let $J_\psi = \text{Jac}(X_\psi)$ there exists a surjective homomorphism defined over \mathbb{Q} (see [YZZ12, §1.2])

$$\pi_f: J_\psi \longrightarrow A_f,$$

where A_f is the modular abelian variety attached to f . Recall that A_f , in this case, is isogenous to $A = \text{Res}_{F/\mathbb{Q}} E$, so that up to composing with an isogeny defined over \mathbb{Q} we have a canonical projection map

$$\pi_E: A_f \longrightarrow E$$

which is defined over F .

Each $D \in \text{Div}^0 \text{CM}_\psi(c) \subset \text{Div}^0(X_\psi)$ gives rise to a point in the Jacobian variety J_ψ , that we also denote by D by slightly abuse of notation. As we remarked in Section 4, Shimura’s reciprocity law implies that $D \in J_\psi(L_c)$. Since π_f is defined over \mathbb{Q} we see obtain the point

$$(5.1) \quad \pi_f(D) \in A_f(L_c).$$

By Lemma 3.5 we have that L_c contains L . Then we define

$$P_{A_f, L}(D) = \text{Tr}_{L_c/L}(\pi_f(D)) \in A_f(L).$$

This trace can be computed analytically as follows. If we let $C_L = \text{rec}^{-1}(\text{Gal}(L_c/L))$ then

$$P_{A_f, L}(D) = \sum_{x \in C_L} \pi_f(x \star D).$$

Now let τ_M denote the element in $\text{Gal}(\mathcal{M}/\mathbb{Q})$ whose fixed field is M . It is easy to check that the point defined as

$$P_{A_f, M}(D) = P_{A_f, L}(D) + \tau_M(P_{A_f, L}(D))$$

belongs to $A_f(M)$. Finally we define

$$(5.2) \quad P_D = \pi_E(P_{A_f, M}(D)) \in E(M).$$

One of the main motivations for the construction of the point P_D is that it extends the construction of [DRZ12] to the case $N^- > 1$. However, a nice feature of the setting considered in [DRZ12] is that in that case the points can be effectively computed (cf. the explicit formula of [DRZ12, Theorem 4.6]) as suitable integrals of the classical modular form f . In our situation, however, the equivalent computation seems to be more difficult, because the modular forms involved are *quaternionic modular forms*. This is the issue that we address in the next paragraph. As we will see in §6, the effective computation of P_D can be accomplished by using p -adic methods.

5.2. Complex uniformization and Heegner points. The projection map π_f is given by a generalization of the classical Eichler–Shimura construction (cf. [Dar04, §4]). In this context, the quaternionic modular form g gives rise to a differential form $\omega_g \in H^0(X_\psi, \Omega^1)$. Let \bar{g} denote the Galois conjugate of g , that is to say, the form attached by Jacquet–Langlands to the Galois conjugate \bar{f} of f . Let

$$\omega_g = 2\pi i g(z) dz \quad \text{and} \quad \omega_{\bar{g}} = 2\pi i \bar{g}(z) dz$$

be the differential forms on \mathcal{H} attached to g and \bar{g} . Let $\Phi = \Phi_{N^+, N^-}$ be the map

$$\begin{aligned} \Phi: \quad \text{Div}^0(\mathcal{H}) &\longrightarrow \mathbb{C} \times \mathbb{C} \\ z_2 - z_1 &\longmapsto \left(\int_{z_1}^{z_2} \omega_g, \int_{z_1}^{z_2} \omega_{\bar{g}} \right). \end{aligned}$$

The subgroup generated by the images under Φ of divisors which become trivial in $\Gamma_\psi \backslash \mathcal{H}$ is a lattice $\Lambda_g \subset \mathbb{C} \times \mathbb{C}$, and \mathbb{C}^2/Λ_g is isogenous to $A_f(\mathbb{C})$. This gives the following analytic description of π_f :

$$(5.3) \quad \begin{aligned} \Phi: \quad \text{Div}^0(\mathcal{H}/\Gamma_\psi) &\longrightarrow A_f(\mathbb{C}) \\ z_2 - z_1 &\longmapsto \left(\int_{z_1}^{z_2} \omega_g, \int_{z_1}^{z_2} \omega_{\bar{g}} \right). \end{aligned}$$

Suppose that $D = \tau_2 - \tau_1 \in \text{Div}^0 \text{CM}_\psi(c)$. We see that the point $\pi_f(D) \in A_f(L_c)$ of (5.1) is given, in complex analytic terms, by the formula

$$(5.4) \quad \pi_f(D) = \left(\int_{\tau_1}^{\tau_2} \omega_g, \int_{\tau_1}^{\tau_2} \omega_{\bar{g}} \right) \in \mathbb{C}^2/\Lambda_g \simeq A_f(\mathbb{C}).$$

The effective computation of the above integrals, however, turns out to be difficult in general when \mathcal{B} is a division algebra, because the newforms in $S_2(\Gamma_\psi)$ cannot be expressed as a Fourier expansion at the cusps. In the next section, and modeling on the classical case of newforms in $S_2(\Gamma_0)$, we will see that the points P_D defined in (5.2) can be computed via p -adic uniformization, instead of complex uniformization.

6. p -ADIC UNIFORMIZATION AND CM POINTS

If p is a prime dividing N^- the abelian varieties $J_0 = \text{Jac}(X_0)$ and $J_\psi = \text{Jac}(X_\psi)$ admit rigid analytic uniformizations at p . That is to say, there exist free groups of finite rank $\Lambda_0, S_0, \Lambda_\psi, S_\psi$ together with isomorphisms

$$(6.1) \quad J_0(\mathbb{C}_p) \simeq \text{Hom}(S_0, \mathbb{C}_p^\times)/\Lambda_0, \quad J_\psi(\mathbb{C}_p) \simeq \text{Hom}(S_\psi, \mathbb{C}_p^\times)/\Lambda_\psi.$$

In this section we use the p -adic uniformization of Čerednik–Drinfel’d, in the explicit formulation provided by Bertolini–Darmon, in order to give a p -adic analytic formula for the points $\pi_f(D) \in A_f(L_c)$ of (5.1). The main feature of this formula, in contrast with that of (5.4), is that it is well suited for numerical computations, thanks to the explicit algorithms of [FM].

6.1. Čerednik–Drinfel’d uniformization. The main reference for this part is [BC91, §5]. Recall the indefinite quaternion algebra \mathcal{B}/\mathbb{Q} of discriminant N^- and $\mathcal{R}_0 \subset \mathcal{B}$ the Eichler order of level N^+ that we fixed in §4. Now let B/\mathbb{Q} be the definite quaternion algebra obtained from \mathcal{B} by interchanging the invariants p and ∞ . That is to say, its set of ramification primes is

$$\text{ram}(B) = \{\ell: \ell \neq p \text{ and } \ell \mid N^-\} \cup \{\infty\}.$$

For every $\ell \mid pN^+$ fix an isomorphism

$$i_\ell: B \otimes \mathbb{Q}_\ell \longrightarrow M_2(\mathbb{Q}_\ell).$$

Let R_0 be a $\mathbb{Z}[\frac{1}{p}]$ -Eichler order of level N^+ in B , which is unique up to conjugation by elements in B^\times . In fact, we can choose a R_0 in such a way that is locally isomorphic to \mathcal{R}_0 at every prime $\ell \neq p$. Let $\Gamma_0^{(p)} = (R_0)_1^\times$ denote the group of norm 1 units, and let

$$R_\psi = \{\gamma \in R_0 : \gamma \in \ker(\psi \circ \eta)\},$$

where $\eta: R_0 \rightarrow \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+}$ denotes the map that sends γ to the upper left entry of $i_{N^+}(\gamma)$ taken modulo N^+ . Set $\Gamma_\psi^{(p)} = (R_\psi)_1^\times$.

Both groups $\Gamma_0^{(p)}$ and $\Gamma_\psi^{(p)}$ act on the p -adic upper half plane \mathcal{H}_p by means of i_p , and the quotients $\Gamma_0^{(p)} \backslash \mathcal{H}_p$ and $\Gamma_\psi^{(p)} \backslash \mathcal{H}_p$ are rigid analytic varieties. In the following statement we collect some particular cases of the Čerednik–Drinfel’d theorem. We denote by \mathbb{Q}_{p^2} (resp. \mathbb{Q}_{p^4}) the unramified extension of \mathbb{Q}_p of degree 2 (resp. of degree 4).

Theorem 6.1. (1) $X_0 \otimes \mathbb{Q}_{p^2} \simeq \Gamma_0^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^2})$.
 (2) If p is split in F then $X_\psi \otimes \mathbb{Q}_{p^2} \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^2})$.
 (3) If p is inert in F then $X_\psi \otimes \mathbb{Q}_{p^4} \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^4})$.

Proof. Part (1) is well know. As for parts (2) and (3), it follows from the Čerednik–Drinfel’d theorem that $X_\psi \otimes \mathbb{C}_p \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p$. The only thing that we need to check is that the isomorphism takes place after extending scalars to \mathbb{Q}_{p^2} if p splits in F , and after extending scalars to \mathbb{Q}_{p^4} if p is inert in F . This follows from the discussion in [BC91, Remark 3.5.3.1]. Indeed, observe that $i_p(R_\psi) \subset M_2(\mathbb{Q}_p)$, contains $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ if $\psi(p) = 1$ (i.e., if p splits in F), but only contains $\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix}$ if $\psi(p) = -1$ (i.e., if p is inert in F). By [BC91, Remark 3.5.3.1] the curve $\Gamma_\psi^{(p)} \backslash \mathcal{H}_p$ and the isomorphism to X_ψ are defined over \mathbb{Q}_{p^2} and over \mathbb{Q}_{p^4} , respectively. \square

6.2. Explicit p -adic uniformization. The main reference for this part is [Dar04, §5]. Let Γ be either $\Gamma_0^{(p)}$ or $\Gamma_\psi^{(p)}$. The group Γ acts on $\mathcal{H}_p = \mathbb{C}_p \backslash \mathbb{Q}_p$ with compact quotient. Therefore, we can speak of $S_2(\Gamma)$, the space of *rigid analytic modular forms of weight 2 on Γ* . It is the set of all rigid analytic functions $h: \mathcal{H}_p \rightarrow \mathbb{C}_p$ such that

$$h(\gamma \cdot \tau) = (c\tau + d)^2 h(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let $\text{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)$ denote the space of \mathbb{C}_p -valued measures of $\mathbb{P}^1(\mathbb{Q}_p)$ with total measure 0. The group Γ acts on it in the following way: if $\mu \in \text{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)$ and $\gamma \in \Gamma$ then $(\gamma \cdot \mu)(U) = \mu(\gamma^{-1}U)$. The Schneider–Teitelbaum theorem gives an isomorphism

$$(6.2) \quad S_2(\Gamma) \simeq \text{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)^\Gamma,$$

where the superscript denotes the elements fixed by Γ .

Let \mathcal{T} denote the Bruhat–Tits tree of $\text{PGL}_2(\mathbb{Q}_p)$. Its set of vertices $\mathcal{V}(\mathcal{T})$ is identified with the set of homothety \mathbb{Z}_p -lattices in \mathbb{Q}_p^2 . Its set of directed edges $\mathcal{E}(\mathcal{T})$ consists of ordered pairs of vertices (v_1, v_2) that can be represented by lattices Λ_1, Λ_2

such that $\Lambda_1 \subset \Lambda_2$ with index p . For a vertex $e = (v_1, v_2)$, we denote $\bar{e} = (v_2, v_1)$, $s(e) = v_1$, and $t(e) = v_2$. An *harmonic cocycle* is a function

$$h: \mathcal{E}(\mathcal{T}) \longrightarrow \mathbb{C}_p$$

such that $h(e) = -h(\bar{e})$ for all $e \in \mathcal{E}$, and such that for all $v \in \mathcal{V}(\mathcal{T})$

$$\sum_{s(e)=v} h(e) = 0.$$

The group Γ acts on \mathbb{Q}_p^2 via i_p , and this induces an action on $\mathcal{E}(\mathcal{T})$. The space of Γ -invariant measures can be identified with the set Γ -invariant of harmonic cocycles. This gives an integral structure

$$\text{Meas}_0(\mathbb{P}_1(\mathbb{Q}_p), \mathbb{Z})^\Gamma \subset \text{Meas}_0(\mathbb{P}_1(\mathbb{Q}_p), \mathbb{C}_p)^\Gamma$$

given by the \mathbb{Z} -valued harmonic cocycles. It thus gives rise, via the isomorphism (6.2), to an integral structure $S_2(\Gamma, \mathbb{Z}) \subset S_2(\Gamma)$.

If $\mu \in \text{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})$ and r is a continuous function on $\mathbb{P}^1(\mathbb{Q}_p)$ then the *multiplicative integral* of r against μ is defined as

$$\oint_{\mathbb{P}^1(\mathbb{Q}_p)} r(t) d\mu(t) = \lim_{\{U_a\}} \prod r(t_a)^{\mu(U_a)},$$

where the limit is defined over increasingly fine disjoint covers $\{U_a\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ and $t_a \in U_a$ is any sample point.

If $h \in S_2(\Gamma, \mathbb{Z})$ and $z_1, z_2 \in \mathcal{H}_p$ the *multiplicative line integral* $\oint_{z_1}^{z_2} h(z) dz$ is defined to be

$$\oint_{z_1}^{z_2} h(z) dz := \oint_{\mathbb{P}^1(\mathbb{Q})} \left(\frac{t - z_1}{t - z_2} \right) d\mu_h(t),$$

where μ_h is the measure attached to h by the Schneider–Teitelbaum isomorphism (6.2). This is used to define the p -adic Abel–Jacobi map

$$\begin{aligned} \Phi_{AJ}: \quad \text{Div}^0(\mathcal{H}_p) &\longrightarrow \text{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times) \simeq (\mathbb{C}_p^\times)^g \\ z_1 - z_2 &\longmapsto \left(h \mapsto \oint_{z_1}^{z_2} h(z) dz \right), \end{aligned}$$

where g denotes the genus of $\Gamma \backslash \mathcal{H}_p$. The group of degree 0 divisors in \mathcal{H}_p that become trivial on $\Gamma \backslash \mathcal{H}_p$ are mapped by Φ_{AJ} to a lattice $\Lambda_\Gamma \subset \text{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times)$. This gives

$$\phi_{AJ}: \quad \text{Div}^0(\Gamma \backslash \mathcal{H}_p) \longrightarrow \text{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times) / \Lambda_\Gamma \simeq \text{Jac}(X_\Gamma)(\mathbb{C}_p).$$

By particularizing this to the groups $\Gamma_0^{(p)}$ and $\Gamma_\psi^{(p)}$ one obtains an explicit expression for the rigid analytic uniformizations of (6.1):

$$\text{Div}^0(\Gamma_0^{(p)} \backslash \mathcal{H}_p) \simeq J_0(\mathbb{C}_p) \xrightarrow{\Phi_{AJ}} \text{Hom}(S_2(\Gamma_0^{(p)}, \mathbb{Z}), \mathbb{C}_p^\times) / \Lambda_0,$$

$$\text{Div}^0(\Gamma_\psi^{(p)} \backslash \mathcal{H}_p) \simeq J_\psi(\mathbb{C}_p) \xrightarrow{\Phi_{AJ}} \text{Hom}(S_2(\Gamma_\psi^{(p)}, \mathbb{Z}), \mathbb{C}_p^\times) / \Lambda_\psi.$$

6.3. CM points and the p -adic uniformization. Let $\mathrm{CM}_0^p(c) \subset \Gamma_0^{(p)} \backslash \mathcal{H}_p$ (resp. $\mathrm{CM}_\psi^p(c) \subset \Gamma_\psi^{(p)} \backslash \mathcal{H}_p$) denote the set of points corresponding to $\mathrm{CM}_0(c) \subset X_0$ (resp. $\mathrm{CM}_\psi(c) \subset X_\psi$) under the isomorphism $X_0(\mathbb{C}_p) \simeq \Gamma_0 \backslash \mathcal{H}_p$ (resp. $X_\psi(\mathbb{C}_p) \simeq \Gamma_\psi \backslash \mathcal{H}_p$).

Bertolini and Darmon give in [BD98] an explicit description of $\mathrm{CM}_0^p(c)$ in terms of certain optimal embeddings of the order of conductor c into B . Next, we use this in order to derive the corresponding description of $\mathrm{CM}_\psi^p(c)$.

Let R_0 be an Eichler order of B of level N^+ as in §6.1. Let $\varphi: \mathcal{O}_c[\frac{1}{p}] \hookrightarrow R_0$ be an optimal embedding of $\mathbb{Z}[\frac{1}{p}]$ -algebras. It has a single fixed point $\tau_\varphi \in \mathcal{H}_p$ satisfying

$$\alpha \begin{pmatrix} \tau_\varphi \\ 1 \end{pmatrix} = i_p(\varphi(\alpha)) \begin{pmatrix} \tau_\varphi \\ 1 \end{pmatrix}$$

for all $\alpha \in \mathcal{O}_c[\frac{1}{p}]$. As before, we can define the notion of *normalized embedding*: the isomorphism

$$i_{N^+}: R_0 \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_{N^+}): c \in N^+ \mathbb{Z}_{N^+} \right\}.$$

allows us to define the homomorphism

$$\eta: R_0 \longrightarrow \mathbb{Z}/N^+ \mathbb{Z}$$

that sends each element x to the upper left entry of i_{N^+} modulo N^+ . Then we say that an optimal embedding $\varphi: \mathcal{O}_c[\frac{1}{p}] \hookrightarrow R_0$ is normalized with respect to \mathfrak{N}^+ if $\ker(\eta \circ \varphi) = \mathfrak{N}^+$. The explicit description of $\mathrm{CM}_0^p(c)$ given by Bertolini–Darmon is then:

$$\mathrm{CM}_0^p(c) = \{[\tau_\varphi] \in \Gamma_0^{(p)} \backslash \mathcal{H}_p: \varphi \in \mathcal{E}(c, R_0)\}.$$

Therefore, the set $\mathrm{CM}_\psi^p(c)$ is given by:

$$\mathrm{CM}_\psi^p(c) = \{[\tau_\varphi] \in \Gamma_\psi^{(p)} \backslash \mathcal{H}_p: \varphi \in \mathcal{E}(c, R_0)\}.$$

As a consequence of Proposition 4.2 we see that $\Phi_{\mathrm{AJ}}(\mathrm{Div}^0(\mathrm{CM}_\psi^p(c)))$ is contained in $J_\psi(L_c)$.

6.4. A p -adic analytic formula for ATR points on \mathbb{Q} -curves. Recall the modular form $f \in S_2(\Gamma_0(N), \psi)$ corresponding to E . There exists a rigid analytic modular form $h \in S_2(\Gamma_\psi, \mathbb{C}_p)$ which is an eigenvector for the Hecke operators, and has the same system of eigenvalues as f . Since the eigenvalues of f are defined over the quadratic imaginary field K_f we can identify h with an harmonic cocycle with values in the ring of integers of K_f , and we denote by \bar{h} the complex conjugated cocycle. Then $h_0 := (h + \bar{h})/2$ and $h_1 := (h - \bar{h})/2i$ belong to $S_2(\Gamma_\psi, \mathbb{Z})$. Let $\Phi^{(p)}$ be the map

$$\begin{aligned} \Phi^{(p)}: \quad \mathrm{Div}^0(\mathcal{H}_p) &\longrightarrow \mathbb{C}_p^\times \times \mathbb{C}_p^\times \\ z_2 - z_1 &\longmapsto \left(\int_{z_1}^{z_2} h_0(z) dz, \int_{z_1}^{z_2} h_1(z) dz \right). \end{aligned}$$

The image of the divisors whose image under $\Phi^{(p)}$ becomes trivial in $\Gamma_\psi^{(p)} \backslash \mathcal{H}_p$ generates a lattice $\Lambda_f^p \subset \mathbb{C}_p^\times \times \mathbb{C}_p^\times$, and the quotient $\mathbb{C}_p^\times \times \mathbb{C}_p^\times / \Lambda_f^p$ is isogenous to $A_f(\mathbb{C}_p)$. In particular, if $D = \tau_2 - \tau_1 \in \mathrm{Div}^0 \mathrm{CM}_0^p(c)$ we find the following p -adic analytic formula for the corresponding CM point in A_f :

$$(6.3) \quad \pi_f(D) = \left(\int_{\tau_1}^{\tau_2} h_0(z) dz, \int_{\tau_1}^{\tau_2} h_1(z) dz \right),$$

which in fact belongs to $A_f(L_c)$. The above formula for $\pi_f(D)$ can be explicitly computed, thanks to a slight modification of the explicit algorithms of [FM]. In the next section we give a detailed example on how these algorithms can be used in order to compute in practice $\pi_f(D)$, and therefore also the point $P_D \in E(M)$.

7. AN EXAMPLE

The goal of this section is to illustrate with an example the construction carried out above. Let $F = \mathbb{Q}(\sqrt{5})$ and consider the elliptic curve defined over F given as

$$E: y^2 = x^3 + (-432\sqrt{5} - 1296)x + (-113184\sqrt{5} - 282960).$$

Remark that E is a \mathbb{Q} -curve, and has conductor $39\mathcal{O}_F$. We will take $p = 13$. The modular form f_E attached to E belongs to $S_2(135, \psi)$, where ψ is the unique quadratic character $\psi: (\mathbb{Z}/5\mathbb{Z})^\times \rightarrow \{\pm 1\}$ of conductor 5. Note that the form f_E has field of coefficients $\mathbb{Q}(\sqrt{-1})$.

We need to construct the quotient of the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the group Γ_ψ . In order to do so, the algorithms of [FM] have been adapted to work with congruence subgroups such as Γ_ψ . The main algorithm of [FM] returns, given two vertices or edges of the Bruhat-Tits tree, the (possibly empty) set of elements of Γ_0 relating them. One just needs to check whether the intersection of this set with Γ_ψ is empty, which is easily done. The quotient graph that we obtain is represented in Figure 1. It consists of 4 vertices and 24 edges. The numbers next to each side of the square in Figure 1 describe how many edges link each of the two corresponding vertices. For example, there are 8 edges connecting v_0 with v_1 . Note that all vertices have valency $14 = p + 1$, so all of them have trivial stabilizers.

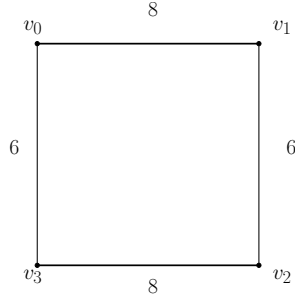


FIGURE 1. Quotient $\Gamma_\psi \backslash \mathcal{E}(\mathcal{T})$

The space of harmonic cocycles on $\Gamma_\psi \backslash \mathcal{E}(\mathcal{T})$ has dimension 25. Taking the common eigenspace on which T_{19} acts as 4 and T_2 acts as 0 we obtain a 2-dimensional subspace associated to f_E . An integral basis of this subspace is given by the harmonic cocycles h_0 and h_1 , which we proceed to describe. The harmonic cocycle h_0 has support on four edges, and takes values in ± 1 there. In fact, it takes the value $+1$ and -1 once on two edges connecting v_0 and v_3 , and the value $+1$ and -1 on two edges connecting v_1 and v_2 . The harmonic cocycle h_1 can be described exactly as h_0 , but they have disjoint supports.

Moreover, T_3 satisfies:

$$T_3(h_0) = -h_1 \text{ and } T_3(h_1) = h_0.$$

Define also $\alpha = 2\sqrt{5} - 1$, and let $M = F(\sqrt{\alpha})$, which is ATR. In this case, the resulting field is $K = \mathbb{Q}(\sqrt{-19})$, which has class number 1. Let g be a root in \mathbb{C}_p of the polynomial $x^2 - x + 5$, and let

$$\tau = (6g+1) + (8g+12)13 + (7g+11)13^2 + (3g+3)13^3 + (12g+9)13^4 + (6g+1)13^5 + \dots$$

be a fixed point under an embedding φ of the maximal order of K into the Eichler order $R_0(1)$ of the quaternion algebra $B = (-3, -1)$, having basis:

$$R_0(1) = \langle 1, j, 5/2j + 5/2k, 1/2 + 1/2i - 3/2j - 3/2k \rangle.$$

We consider the divisor $D = (\tau) - (\bar{\tau})$ and calculate:

$$\begin{aligned} J_0 = \int_{\tau}^{\bar{\tau}} \omega_{h_0} = & (8g+12) + (3g+1)13 + (7g+10)13^2 + (8g+8)13^3 + (7g+1)13^4 + \\ & (7g+6)13^5 + (9g+8)13^6 + (7g+7)13^7 + (4g+9)13^8 + (4g+4)13^9 + \\ & (5g+12)13^{10} + (8g+1)13^{11} + (11g+11)13^{12} + \dots \end{aligned}$$

and in fact $J_1 = J_0$.

We calculate the image of J_0 under the Tate uniformization map, to get coordinates $(x, y) \in E(\mathbb{C}_p)$:

$$\begin{aligned} x = & (12h^3 + 3h^2 + 4h + 1) + (9h^3 + 10h^2 + h + 9)13 + (6h^3 + 5h^2 + 3h + 9)13^2 + \\ & (6h^3 + 8h^2 + 8)13^3 + (8h^3 + 2h^2 + 5h + 8)13^4 + (4h^3 + 9h^2 + 4h + 6)13^5 + \dots \end{aligned}$$

and

$$\begin{aligned} y = & (11h^3 + 5h^2 + 2h + 9) + (12h^3 + 12h^2 + h + 10)13 + (7h^2 + 10h + 7)13^2 + \\ & (2h^3 + 5h^2 + 9h + 7)13^3 + (5h^3 + 2h^2 + 4h + 4)13^4 + (3h^3 + 3h + 11)13^5 + \dots \end{aligned}$$

Here, h satisfies:

$$h^4 + 3h^2 + 12h + 2 = 0.$$

We have carried out all the calculations to precision 13^{80} , and up to this precision it turns out that x is a root of the irreducible polynomial:

$$P_x(T) = T^4 + 60T^3 + 19728T^2 + 380160T + 40144896$$

and y is a root of the irreducible polynomial:

$$\begin{aligned} P_y(T) = & T^8 - 1166400T^6 + 5027006707200T^4 - 321342050396160000T^2 + \\ & 75899706935371407360000. \end{aligned}$$

The polynomial $P_y(T)$ factors as two quartics over F . We let \mathcal{M}/F be the quartic extension generated by one of these two factors, and we remark that $P_y(T)$ splits completely over \mathcal{M} , so it is actually the splitting field of $P_y(T)$. Let α be a root of $P_y(T)$ in \mathcal{M} . Then the coordinates (x, y) are defined over \mathcal{M} and correspond to the point:

$$((1/12960\sqrt{5} - 1/4320)\alpha^2 + 3/2\sqrt{5} + 15/2, \alpha) \in E(\mathcal{M}).$$

Since \mathcal{M} contains the field M , we can compute the trace of this point down to M , to obtain the point of infinite order

$$P_D = \left(\frac{474\sqrt{5} + 750}{19}, \frac{20412\sqrt{5} + 19440}{361}\sqrt{\alpha} \right) \in E(M).$$

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INSTITUT FÜR EXPERIMENTELLE MATHEMATIK, ESSEN

E-mail address: xevi.guitart@gmail.com

UNIVERSITY OF WARWICK, WARWICK

E-mail address: m.masdeu@warwick.ac.uk